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1980 J. Phys. A: Math. Gen. 13 2807

(http://iopscience.iop.org/0305-4470/13/8/028)

J. Phys. A: Math. Gen. 13 (1980) 2807–2816. Printed in Great Britain

On the conformal invariance in quantum electrodynamics

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Received 7 November 1979

Abstract. A realisation of the conformal group with operator-valued dimensions is found which leaves invariant the Dirac equation for the spinor electrodynamics with massless fermions. All conformally invariant two- and three-point Wightman functions are calculated and it is shown that they are in agreement with the equation.

1. Introduction

In the paper of Sotkov *et al* (1979) it was shown that the massless Dirac equation with electromagnetic interaction

$$\mathrm{i}\gamma^{\mu}\partial_{\mu}\psi = e\!:\!A_{\mu}\gamma^{\mu}\psi\!: \qquad (\mu = 0, 1, 2, 3) \tag{1.1}$$

(where A_μ is the electromagnetic vector-potential and e is the dimensionless charge, $e^2 = \alpha$) is conformally covariant with respect to at least two different representations of the conformal group. The first of them is the well known linear representation for the fields A_μ and ψ, according to which both these fields have canonical conformal dimensions. The second is not standard at all. It is nonlinear, and in addition the conformal dimension of the field ψ, being an operator, is in fact not determined. Both these representations determine, up to multiplicative factors, all of the two- and three-point Wightman functions of the fields A_μ and ψ, provided the vacuum is conformally invariant. The corresponding expressions for these functions are different for the two different representations. As is well known, in the first case the Wightman functions of the fields A_μ and ψ coincide with the corresponding free fields functions. This means that one obtains only trivial solutions of equation (1.1) ($A_\mu = 0$) in that case.

The second case is treated in detail in the paper by Sotkov *et al* (1979) (to be referred to as I). It is shown there that the resulting two- and three-point Wightman functions correspond to solutions of equation (1.1) that are pure gauges.

In the present paper we find a third representation of the conformal group for the fields A_μ and ψ, with respect to which equation (1.1) is invariant. Using the method introduced in I, we obtain the two- and three-point Wightman functions and then show that they are compatible with the equation. Further, it appears that these functions correspond to a certain solution of equation (1.1) for which the vector potential contains transversal terms. Thus the two- and three-point Wightman functions obtained in the present paper give an example for the possible exact expressions for these functions in the massless spinor electrodynamics.

2. Conformal transformations for the electromagnetic potentials

The analysis of the conformal invariance of the massless quantum Thirring model (see Hadjiivanov *et al* 1979) shows that the field operators transform according to a certain continually reducible representation of the two-dimensional conformal group. The operator-valued conformal dimensions were introduced for the first time in this paper. The four-dimensional gradient model of the quantum electrodynamics in this respect is quite analogous to the Thirring model (see I). Here again the leading role among the representations with operator-valued dimensions is played by the nonlinear representations of the conformal group, in four-dimensional space–time in this case. Conventionally, we call basis representations those of the representations that arise in the homogeneous space of the conformal group. The transformations for the scalar field $S(x)$ and the vector field $A_\mu(x)$ discussed in I are basic nonlinear realisations. We briefly recall some of the main features of their structure.

The field $S(x)$, being a Lorentz scalar, transforms non-homogeneously under the action of the scale and special conformal transformations. The equation invariant with respect to these transformations is

$$\Box^2 S(x)=0 \tag{2.1}$$

where $\Box$ is the D'Alembert operator. If $S^{\pm}(x)$ are the frequency parts of $S(x)$, then the commutator

$$[S^+(x), S^-(y)] = -\mathrm{i}\lambda E^+(x-y) \tag{2.2}$$

where

$$E^+(x)=\mathrm{i}(4\pi)^{-2}\ln(-\mu^2x^2-\mathrm{i}0x_0)$$

and μ is an arbitrary constant with the dimension of a mass. In terms of the frequency parts the discussed transformations have the following form:

$$U_D(r)S^{\pm}(x)U_D^{-1}(r)=S^{\pm}(rx)+eq^{\pm}\ln r \tag{2.3}$$

$$U_K(\alpha)S^{\pm}(x)U_K^{-1}(\alpha)=S^{\pm}(x^{(K)})-eq^{\pm}\ln|\rho(\alpha,x)| \tag{2.4}$$

where $U_D(r)$ and $U_K(\alpha)$ are the operators of the representations of the scale (with parameter r) and the special conformal (with parameter $\alpha\equiv(\alpha_\mu)$, $\mu=0,1,2,3$) transformation respectively and

$$x_\mu^{(K)}=(x_\mu+x^2\alpha_\mu)/\rho(\alpha,x), \qquad \rho(\alpha,x)=1+2(\alpha x)+\alpha^2x^2. \tag{2.5}$$

As was shown in I, the quantities $q^{\pm}$ are constant operators related to $S(x)$ in the following way:

$$q^+=(q^-)^*=-\frac{\mathrm{i}}{(4\pi)^2e}\lim_{\substack{x^2<0\\|x|\to\infty}}\frac{S^+(x)}{E^+(x)} \tag{2.6}$$

where e is the invariant charge.

Formula (2.6) gives the possibility to calculate the commutators of the operators q^+ and q^- with the fields $S^{\pm}(x)$ and the commutator of q^+ with q^-:

$$[q^{\pm}, S^{\mp}(x)] = \mp\frac{\lambda}{(4\pi)^2e}=\mp\frac{e}{\varkappa} \qquad \varkappa=\frac{(4\pi e)^2}{\lambda}$$

$$[q^{\pm}, S^{\pm}(x)]=[q^+,q^-]=0. \tag{2.7}$$

Equations (2.7) being obtained, one can immediately check the invariance of equations (2.1) and (2.2) with respect to the transformations (2.3) and (2.4).

The field $A_\mu(x)$ is a vector field and its scale and special conformal transformations are also non-homogeneous. However, these transformations can be obtained from (2.3) and (2.4) by differentiating the latter equations, since $A_\mu(x)$ and $\partial_\mu S(x)$ transform equivalently. That is why the conformally invariant two- and three-point Wightman functions of the field $A_\mu(x)$ are pure longitudinal. It is the field $A_\mu(x)$ that is used in I in order to obtain a conformally covariant description of the gradient model of quantum electrodynamics.

However, besides the basis representations there are other nonlinear realisations of the conformal group. They arise from the linear representations through redetermination of the linearly transforming fields with local transformations belonging to the same group (for these transformations see e.g. Salam and Strathdee (1969) for the conformal group and Ogievetsky (1973)† for the general case). That is why we will conventionally call such nonlinear realisations derivative realisations.

Let a vector field $h_\mu(x)$ with canonical conformal dimension be given, i.e.

$$U_D(r)h_\mu(x)U_D^{-1}(r) = rh_\mu(rx) \tag{2.8}$$

$$U_R h_\mu(x)U_R^{-1} = (\partial x^{(R)\rho}/\partial x^\mu)h_\rho(x^{(R)}). \tag{2.9}$$

Here and in the following we shall use the R-inversion instead of the special conformal transformations. That is why in equation (2.9) U_{R} denotes the operator of the representation of the R-inversion, and

$$x^{(R)\mu} = -x^\mu/x^2. \tag{2.10}$$

Consider the quantity

$$S^\pm(x)h_\mu(x). \tag{2.11}$$

They are again vector fields, but transform according to a derivative nonlinear realisation of the conformal group. We suppose that the vector potentials of the electromagnetic field transform according to this realisation of the conformal group.

If we forget the concrete expression (2.11) we can think that the electromagnetic vector-potential can be decomposed into two parts, $A_\mu^+(x)$ and $A_\mu^-(x)$, that transform with respect to the conformal group in the following way:

$$U_D(r)A_\mu^\pm(x)U_D^{-1}(r) = rA_\mu^\pm(rx) + eq^\mp r \ln rh_\mu(rx) \tag{2.12}$$

$$U_R A_\mu^\pm(x)U_R^{-1} = \frac{\partial x^{(R)\nu}}{\partial x^\mu}[A_\nu^\pm(x^{(R)}) - eq^\pm \ln|x^2|h_\mu(x^{(R)})] \tag{2.13}$$

(it is evident that in particular $S^\pm(x)h_\mu(x)$ transform exactly according to equations (2.12) and (2.13)).

We consider these transformations in the quantised case with the additional assumption that $q^\pm$ and $h_\mu(x)$ commute, while the commutator

$$[q^\pm, A_\mu^\mp(x)] = \mp(e/\varkappa)h_\mu(x). \tag{2.14}$$

We only postulate the latter commutators although one can put forward certain considerations in their favour. We just note that equation (2.14) is implied by equations

† Here one can find further references on the nonlinear realisations of Lie groups.

(2.12) and (2.13) and the assumption of commutativity of $q^{\pm}$ and $h_\mu(x)$. (Of course $q^{\pm}$ are invariant operators here, as in the case of the realisation (2.3) and (2.4)).

Using equation (2.14) one can write down equations (2.12) and (2.13) in the following form:

$$U_D(r)A^{\pm}_\mu(x)U_D^{-1}(r) = r^{1\pm\varkappa q^+q^-}A^{\pm}_\mu(rx)r^{\mp\varkappa q^+q^-} \tag{2.15}$$

$$U_R A^{\pm}_\mu(x)U_R^{-1} = \frac{\partial x^{(R)\nu}}{\partial x^\mu}|x^2|^{\mp\varkappa q^+q^-}A^{\pm}_\nu(x^{(R)})|x^2|^{\pm\varkappa q^+q^-}. \tag{2.16}$$

In the latter expressions the field $h_\mu(x)$ does not enter explicitly. It is evident also that they coincide with (2.8) and (2.9) if one substitutes there $h_\mu(x)$ instead of $A_\mu(x)$.

3. Conformal transformations of the spinor field

According to our assumptions the electromagnetic potential is transformed by means of equations (2.15) and (2.16) (or equivalently equations (2.12) and (2.13)). If we use these transformations for $A_\mu(x)$ in equation (1.1) and ask for the latter to be invariant we can derive the conformal transformations of the spinor field $\psi(x)$ also. However, before doing this it is necessary to determine the RHS of equation (1.1) in terms of the fields that have already been introduced. In the previous section the signs '±' of the field A_μ were introduced in analogy to the expression (2.11) and their meaning ought to be denoting the frequency parts. If such an assumption is right then

$$:A_\mu(x)\psi(x): = A^+_\mu(x)\psi(x) + \psi(x)A^-_\mu(x). \tag{3.1}$$

The judgement whether this assumption is correct can be uniquely held from the requirement for conformal invariance of the equation. Let us substitute (3.1) into equation (1.1) and make a conformal transformation. In order for equation (3.1) to keep its form invariant, it is necessary to assume the following transformations for the spinor field:

$$U_D(r)\psi(x)U_D^{-1}(r) = r^{3/2+\varkappa q^+q^-}\psi(rx)r^{\varkappa q^+q^-} \tag{3.2}$$

$$U_R\psi(x)U_R^{-1} = |x^2|^{-2-\varkappa q^+q^-}\hat{x}\psi(x^{(R)})|x^2|^{-\varkappa q^+q^-}; \qquad \hat{x} = \gamma^\mu x_\mu \tag{3.3}$$

(up to now the problem for the commutators of $q^{\pm}$ and ψ has not arisen).

Nevertheless, after this procedure equation (1.1) does not obtain its initial form. One can see that there do not exist any meaningful reformulations of equations (2.15) and (2.16) or (3.2) and (3.3) that are capable of keeping equation (1.1) invariant. So we come to the conclusion that equation (3.1) is not correct for the field with the conformal properties that we have introduced.

If, however, instead of (3.1) we take for the definition of the quantity $:A_\mu(x)\psi(x):$ the formula

$$:A_\mu(x)\psi(x): \stackrel{\text{def}}{=} \lim_{x_2\to x_1=x}\Big\{A^+_\mu(x_1)\psi(x_2) + \psi(x_2)A^-_\mu(x_1) - \frac{i\varkappa}{e}\partial_\mu \ln|x_{12}^2|[q^+q^-\psi(x_2) + \psi(x_2)q^+q^-]\Big\} \tag{3.4}$$

where

$$x_{12} = x_1 - x_2$$

then we can immediately show that equation (1.1) is conformally covariant with respect to the transformations (2.15), (2.16), (3.2) and (3.3). The proof of this statement is given in the Appendix. One can see that it is not necessary to commute q^+ and q^- with $\psi(x)$ in the course of this proof. However, the latter commutators are necessary when one calculates the Wightman functions. We postulate them as for the case of the commutators (2.14). Namely, we suppose that

$$[q^{\pm}, \psi(x)] = \omega^{\pm}\psi(x) \tag{3.5}$$

where $\omega^+ = (\omega^-)^*$ (* denotes complex conjugation) is a complex number. In general we write down equation (3.5) in analogy to the corresponding equation for the gradient model. On the other hand, equation (3.5) is the simplest conformally invariant commutator. Besides the commutator (3.5) we have also

$$[q^{\pm}, \bar{\psi}(x)] = -\omega^{\pm}\bar{\psi}(x) \tag{3.6}$$

where $\bar{\psi}(x)$ is the Dirac conjugated field.

Finally we must consider the correlations of the spinor field $\psi(x)$ with the 'bare' field $h_\mu(x)$. Since the latter field is transformed according to the canonical conformal transformation it is evident that it is a free field. That is why we ought to suppose that

$$[\psi(x_1), h_\mu(x_2)] = 0. \tag{3.7}$$

4. Two- and three-point Wightman functions

Consider the vacuum of the theory, based on the quantised equation (1.1) with the definition (3.4), to be conformally invariant:

$$U_D|0\rangle = U_R|0\rangle = |0\rangle. \tag{4.1}$$

Then we can calculate the two- and three-point Wightman functions and check their compatibility with equation (1.1). The method of obtaining these functions is well known and has been already used in I (see also Todorov *et al* (1978)). We omit the details of the calculations and will write down just the equations and their general solutions.

We begin with the two-point function of the fields. We denote this function by

$$\delta_{\mu\nu}(x_{12}) = \langle 0|A_\mu(x_1)A_\nu(x_2)|0\rangle. \tag{4.2}$$

Then the conditions of scale and R invariance read as follows:

$$\begin{aligned}\delta_{\mu\nu}(x_{12}) &= r^2\delta_{\mu\nu}(rx_{12}) + (2e^2\Delta/\varkappa)\ln r\,\partial_\mu\partial_\nu \ln|x_{12}^2| \\ \delta_{\mu\nu}(x_{12}) &= \frac{\partial x_1^{(R)\rho}}{\partial x_1^\mu}\frac{\partial x_2^{(R)\sigma}}{\partial x_2^\nu}\delta_{\rho\sigma}(x_{12}^{(R)}) - \frac{e^2\Delta}{\varkappa}\ln|x_1^2x_2^2|\partial_\mu\partial_\nu \ln|x_{12}^2|\end{aligned} \tag{4.3}$$

where Δ is a normalisation constant of the two-point function of the field $h_\mu(x)$:

$$\delta_{\mu\nu}^{(0)}(x_{12}) \equiv \langle 0|h_\mu(x_1)h_\nu(x_2)|0\rangle = \Delta\partial_\mu\partial_\nu \ln|x_{12}^2|.$$

The non-homogeneous terms in equations (4.3) arise from the commutation of the operators q^+ and q^- with the field $A_\mu(x)$. In this procedure according to equation (2.14) the fields $h_\mu(x)$ occur, thus relating the functions $\delta_{\mu\nu}(x_{12})$ and $\delta_{\mu\nu}^{(0)}(x_{12})$.

The general solution of the functional equations (4.3) has the form

$$\delta_{\mu\nu}(x_{12}) = -(e^2\Delta/\varkappa)\ln|x_{12}^2|\partial_\mu\partial_\nu \ln|x_{12}^2| + C\partial_\mu\partial_\nu \ln|x_{12}^2| \tag{4.4}$$

(C is an arbitrary constant). This function has the remarkable feature that it contains a transversal term. Indeed it is not difficult to show that the function (4.4) can be written in the form

$$\delta_{\mu\nu}(x_{12}) = \frac{e^2\Delta}{\varkappa}\frac{g_{\mu\nu}}{x_{12}^2} + \text{gauge terms.} \tag{4.5}$$

Consider now the two-point function of the spinor field

$$G_{\alpha\beta}(x_{12}) = \langle 0|\bar{\psi}_\beta(x_1)\psi_\alpha(x_2)|0\rangle. \tag{4.6}$$

The requirements for scale and R invariance lead to the standard equations, which is why we do not write them here. We just write down the expression that is obtained for the function (4.6),

$$G(x_{12}) = N\hat{x}_{12}(x_{12}^2)^{-2-\varkappa\omega^+\omega^-} \tag{4.7}$$

where N is an arbitrary constant.

At last one can obtain the three-point function analogously. The requirements for scale and R invariance for the function

$$\Gamma_{\mu,\alpha\beta}(x_1, x_2, x_3) = \langle 0|\bar{\psi}_\beta(x_1)A_\mu^+(x_2)\psi_\alpha(x_3)|0\rangle \tag{4.8}$$

will read in our case

$$\Gamma_\mu(x_1, x_2, x_3) = r^{4+2\varkappa\omega^+\omega^-}\Gamma_\mu(rx_1, rx_2, rx_3) \tag{4.9}$$

$$\Gamma_\mu(x_1, x_2, x_3) = |x_1^2x_2^2|^{-2-\varkappa\omega^+\omega^-}\frac{\partial x_2^{(R)\nu}}{\partial x_2^\mu}\hat{x}_3\Gamma_\nu(x_1^{(R)}, x_2^{(R)}, x_3^{(R)})\hat{x}_1. \tag{4.10}$$

It is evident that these equations are the standard ones and that is why their general solution is well known to be

$$\Gamma_\mu(x_1, x_2, x_3) = (x_{13}^2)^{-1-\varkappa\omega^+\omega^-}\left(C_1\frac{\hat{x}_{13}}{x_{13}^2}\partial_{2\mu}\ln\frac{x_{12}^2}{x_{32}^2} + C_2\frac{\hat{x}_{32}\gamma_\mu\hat{x}_{21}}{x_{32}^2x_{21}^2}\right) \tag{4.11}$$

where C_1 and C_2 are arbitrary constants.

In order to fix these constants we consider the function

$$\tilde{\Gamma}_\mu(x_{12}) = \langle 0|\bar{\psi}(x_1):A_\mu(x_2)\psi(x_2):|0\rangle. \tag{4.12}$$

It is evident that according to the definition (3.4) the latter function should be obtained by means of the following equation:

$$\tilde{\Gamma}_\mu(x_{12}) = \lim_{x_3\to x_2}\left(\Gamma_\mu(x_1, x_2, x_3) - \frac{i\varkappa\omega^+\omega^-}{e}\partial_\mu\ln|x_{23}^2|G(x_{13})\right). \tag{4.13}$$

The condition of existence of a non-zero limit (4.13) gives the possibility of fixing the arbitrary constants C_1 and C_2. Indeed, let us substitute $\Gamma_\mu(x_1, x_2, x_3)$ and $G(x_{13})$ from (4.7) and (4.11), respectively, into (4.13). If we now assume that

$$C_1 = -(i\varkappa\omega^+\omega^-/e)N \qquad \text{while } C_2 = 0 \tag{4.14}$$

then the limit (4.13) exists and

$$\tilde{\Gamma}_\mu(x_{12}) = (2i\varkappa\omega^+\omega^-/e)N\hat{x}_{12}(x_{12})_\mu(x_{12}^2)^{-3-\varkappa\omega^+\omega^-}. \tag{4.15}$$

Finally we can show that the Wightman functions obtained previously in this section are consistent with equation (1.1). For this purpose we denote the argument in equation (1.1) by x_2, then multiply both parts of this equation on the left by $\bar{\psi}(x_1)$ and take the mean value of the so-obtained operator equality. As a result, we have

$$\mathrm{i}\gamma^{\mu}\partial_{2\mu}G(x_{12}) = e\gamma^{\mu}\tilde{\Gamma}_{\mu}(x_{12}). \tag{4.16}$$

It is not difficult to substitute (4.7) and (4.15) into (4.16) and see that it then becomes an identity.

5. Gauge invariance

The gauge invariance of (1.1) gives certain additional relations between the Wightman functions obtained in the previous section. Leaving aside the problem for the possible local gauge transformations that keep equation (1.1) invariant, we consider only those gauges that are related to the field $S(x)$. The latter transformations have the form

$$A^{g}_{\mu}(x) = A_{\mu}(x) + z\,\partial_{\mu}S(x) \tag{5.1}$$

$$\psi^{g}(x) = :\mathrm{e}^{-\mathrm{i}ezS(x)}\psi(x): \equiv \mathrm{e}^{-\mathrm{i}ezS^{+}(x)}\psi(x)\,\mathrm{e}^{-\mathrm{i}ezS^{-}(x)}. \tag{5.2}$$

They are one-parametric and z is their parameter. Now the electromagnetic field $A^{g}_{\mu}(x)$ has a new transformation law with respect to the conformal group. We do not write down these laws, since the Wightman function

$$\delta^{g}_{\mu\nu}(x_{12}) = \langle 0|A^{g}_{\mu}(x_1)A^{g}_{\nu}(x_2)|0\rangle \tag{5.3}$$

can be obtained directly from (5.1), taking into account (2.2) and (4.4). It appears that the function $\delta^{g}_{\mu\nu}(x_{12})$ has the same form as (4.4) but with a different constant C. The new constant has the form

$$C_g = \lambda z^2 + C. \tag{5.4}$$

At the same time, the transformation (5.2) changes the degree of the homogeneous two-point function of the spinor field. Indeed, the conformal transformations of the field $\psi^{g}(x)$ now have the form

$$U_D(r)\psi^{g}(x)U_D^{-1}(r) = r^{3/2+\varkappa q^{+}q^{-}-\mathrm{i}e^2zq^{+}}\psi^{g}(rx)r^{\varkappa q^{+}q^{-}-\mathrm{i}e^2zq^{-}} \tag{5.5}$$

$$U_R\psi^{g}(x)U_R^{-1} = |x^2|^{-2-\varkappa q^{+}q^{-}+\mathrm{i}e^2zq^{+}}\hat{x}\psi^{g}(x^{(R)})|x^2|^{-\varkappa q^{+}q^{-}+\mathrm{i}e^2zq^{-}} \tag{5.6}$$

and

$$[q^{\pm}\psi^{g}(x)] = \omega^{\pm}_{g}\psi^{g}(x) \tag{5.7}$$

where

$$\omega^{\pm}_{g} = \omega^{\pm} \pm \mathrm{i}\lambda\,(4\pi)^{-2}z. \tag{5.8}$$

If we calculate again the Wightman function (4.6), we obtain

$$G^{g}(x_{12}) \equiv \langle 0|\bar{\psi}^{g}(x_1)\psi^{g}(x_2)|0\rangle = \hat{x}_{12}(x^2_{12})^{-2-2\varkappa\omega^{+}_{g}\omega^{-}_{g}-\mathrm{i}e^2z(\omega^{-}_{g}-\omega^{+}_{g})}. \tag{5.9}$$

These results show that the constants that appear in the Wightman functions are related in some way. In particular, this correlation can be taken into account, if we consider the quantities C, $\omega^{\pm}$ and d (the power in formula (5.9)) as functions of the gauge parameter z. Taking into account (5.8), we see that $\omega^{\pm}(z)$ are linear functions of z, and since the additional gauge term in $\omega^{\pm}$ is imaginary, we come to the conclusion that

$$\mathrm{Re}\,\omega^{+}(z)\equiv\sigma_1=\text{constant}. \tag{5.10}$$

Then for

$$\mathrm{Im}\,\omega^{+}(z)\equiv\sigma_2(z)$$

we have

$$\sigma_2(z)=[\lambda/(4\pi)^2]z+B. \tag{5.11}$$

We denote by $d(z)$ the power on the RHS of (5.9), i.e.

$$d(z)=-2-2\varkappa\omega^{+}(z)\omega^{-}(z)+\mathrm{i}e^2z[\omega^{+}(z)-\omega^{-}(z)]. \tag{5.12}$$

Then from (5.10) and (5.11) we obtain

$$\omega^{\pm}(z)=\pm[\mathrm{i}\lambda/(4\pi)^2]z\pm\mathrm{i}B+\sigma_1. \tag{5.13}$$

Substituting (5.13) into (5.12) we obtain

$$d(z)=-2-\frac{32\pi^2}{\lambda}e^2(\sigma_1^2+B^2)-6e^2Bz-\frac{\lambda e^2}{(4\pi)^2}z^2. \tag{5.14}$$

Finally, bearing in mind (5.4), we have

$$C(z)=\lambda z^2+C_0. \tag{5.15}$$

Equalities (5.13), (5.14) and (5.15) show how one must bring into agreement the basic characteristics of the Wightman functions $\omega^{\pm}$, d and C in the case of fixed gauge. Given the constants C_0, σ_1 and B, each value of z corresponds to a certain gauge, and then using equations (5.13), (5.14) and (5.15) one can calculate the mutually consistent values of $\omega^{\pm}$, d and C in this gauge.

Acknowledgments

The authors are grateful to Drs V Petcova and I Todorov for useful discussions and critical remarks.

Appendix

Bearing in mind formula (3.4), equation (1.1) can be written as

$$\mathrm{i}\gamma^{\mu}\partial_{\mu}\psi(x)=e\gamma^{\mu}\lim_{x_2\to x_1=x}\{A_{\mu}^{+}(x_1)\psi(x_2)+\psi(x_2)A_{\mu}^{-}(x_1)$$
$$-\mathrm{i}(\varkappa/e)\partial_{\mu}\ln|x_{12}^2|[q^+q^-\psi(x_2)+\psi(x_2)q^+q^-]\}. \tag{A1}$$

(a) *Proof of scale invariance*

Multiplying equation (A1) from the left by $U_D(r)$ and from the right by $U_D^{-1}(r)$, and taking into account equations (2.15) and (3.2), we have

$$r^{5/2+\varkappa q^+q^-}\mathrm{i}\gamma^{\mu}\partial_{\mu}^{D}\psi(rx)r^{\varkappa q^+q^-}$$
$$=r^{5/2+\varkappa q^+q^-}e\gamma^{\mu}\lim_{x_2\to x_1=x}\Big(A_{\mu}^{+}(rx_1)\psi(rx_2)+\psi(rx_2)A_{\mu}^{-}(rx_1)$$
$$-\mathrm{i}\frac{\varkappa}{e}\partial_{\mu}^{D}\ln|r^2x_{12}^2|[q^+q^-\psi(rx_2)+\psi(rx_2)q^+q^-]\Big)r^{\varkappa 1^+q^-}$$

where $\partial_\mu^D = \partial/\partial r x^\mu$. Cancelling from the left the factor $r^{5/2+\varkappa q^+q^-}$ and from the right the factor $r^{\varkappa q^+q^-}$ we obtain the initial equation at the point $x' = rx$ and this completes the proof.

(b) *Proof of R-invariance of equation* (A1)

We multiply this equation from the left by U_R and from the right by U_R^{-1} and make use of formulae (2.16) and (3.3). After making the necessary cancellations we obtain the transformed equation in the form

$$\begin{aligned} &i\gamma^\mu \hat{x} \frac{\partial x^{(R)\nu}}{\partial x^\mu} \partial_\nu^{(R)} \psi(x^{(R)}) - 2i\varkappa[q^+q^-\psi(x^{(R)}) + \psi(x^{(R)})q^+q^-] \\ &\quad = e\gamma^\mu \hat{x} \lim_{x_2\to x_1 = x} \Big(\frac{\partial x_1^{(R)\nu}}{\partial x_1^\mu} [A_\nu^+(x_1^{(R)})\psi(x_2^{(R)}) + \psi(x_2^{(R)})A_\nu^-(x_1^{(R)})] \\ &\qquad - i(\varkappa/e)\partial_\mu \ln|x_{12}^2| [q^+q^-\psi(x_2^{(R)}) + \psi(x_2^{(R)})q^+q^-]\Big). \end{aligned} \tag{A2}$$

The first term on the RHS of (A2) is obtained through the following sequence of equalities:

$$\begin{aligned} &\lim_{x_1\to x_2 = x} U_R A_\nu^+(x_1)\psi(x_2)U_R^{-1} \\ &\quad = \lim_{x_1\to x_2=x} \frac{\partial x_1^{(R)\nu}}{\partial x_1^\mu} |x_1^2|^{-\varkappa q^+q^-} A_\mu^+(x_1^{(R)}) \left|\frac{x_1^2}{x_2^2}\right|^{\varkappa q^+q^-} \psi(x_2^{(R)})|x_2|^{-2-\varkappa q^+q^-} \\ &\quad = \lim_{x_1\to x_2=x} \frac{\partial x_1^{(R)\nu}}{\partial x_1^\mu} |x_1^2|^{-\varkappa q^+q^-} \sum_{n=0}^{\infty} \frac{\ln^n|x_1^2/x_2^2|}{n!} \\ &\qquad \times A_\mu^+(x_1^{(R)})(\varkappa q^+q^-)^n \psi(x_2^{(R)})|x_2^2|^{-2-\varkappa q^+q^-} \\ &\quad = |x^2|^{-2-\varkappa q^+q^-} \lim_{x_1\to x_2=x} \left[\frac{\partial x_1^{(R)\nu}}{\partial x_1^\mu} A_\mu^+(x_1^{(R)})\psi(x_2^{(R)})\right] |x^2|^{-\varkappa q^+q^-}. \end{aligned}$$

Let us introduce the notation

$$x_{12}^R = x_1^{(R)} - x_2^{(R)}.$$

Then we have the following identity:

$$\partial_\mu \ln|x_{12}^{(R)}| = \frac{\partial x_1^{(R)\nu}}{\partial x_1^\mu} \partial_\nu^{(R)} \ln|(x_{12}^{(R)})^2| + \frac{2x_{1\mu}}{x_1^2}$$

where $\partial_\nu^{(R)}$ denotes the operation $\partial/\partial x^{(R)\nu}$. Substituting the last identity into the second term of the RHS of equation (A2), we obtain

$$i\gamma^\mu \hat{x} \frac{\partial x^{(R)\nu}}{\partial x^\mu} \partial_\nu^{(R)} \psi(x^{(R)}) = e\gamma^\mu \hat{x} \frac{\partial x^{(R)\nu}}{\partial x^\mu} :A_\nu(x^{(R)})\psi(x^{(R)}):. \tag{A3}$$

Finally, it is necessary to make use of one more identity. After cancelling the non-degenerate matrices $\hat{x}/x^2$, we see that equation (A2) coincides with equation (A1) at the point $x^1 = x^{(R)}$.

Thus the scale and special conformal invariance of equation (A1) is proved.

References

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